Elusive Odds in the game of Bridge

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Abstract

In the playing phase of the game of bridge, knowledge of the odds of card distributions is an important aid to find the optimal line of play. Classic probability theory offers a framework for calculating a priori odds for any distribution of cards and also for calculating a posteriori odds, when during play certain evidence has become available. Applying probability theory is sometimes considered as complex and alternative simpler methods have been suggested for calculating these odds such as the *Vacant Places Method*, the *Deletion Principle* and the *Conditional Ratio Shortcut*. We show examples in the bridge literature where the application of these concepts leads to incorrect results. As an alternative we present a RDP (*Right Deletion Principle*) which leads to correct results.

1 Introduction

Developing a playing plan in bridge is an example of decision making under uncertainty. This is a field with roots in mathematics and in management science. There are many types of uncertainty, a main classification being randomness and unknown strategy of opponents. The first type can quantitatively be modelled through probability theory, while the second type requires qualitative models of human psychology. Real-life decision making as in business and warfare contains both types of uncertainty. This is formally true in bridge too, however the randomness of the cards deal is the dominant source of uncertainty and as a consequence probability theory is widely propagated for finding the optimal play.

The basic tool of probability theory in bridge is Bayes' theorem, which provides a rule for updating hypotheses of possible distributions of opponent's cards when evidence i.e. the play of cards by opponents becomes available. Bayes' rule is theoretically well grounded in classic probability theory, however some people find the application of probability theory complicated and prefer simpler rules for the calculation of probabilities of hypotheses. Three of such simplifications are the *Vacant Places Method* for a priori probablities of card distributions, the *Deletion Principle* and the *Conditional Ratio Shortcut* for the updating of these probabilities.

We will discuss under which conditions application of the *Vacant Places Method* is correct. The *Vacant Places Method* is correct when calculating a priori distributions, and in general incorrect when calculating a posteriori distributions. Further we discuss the general structure of deletion concepts for the calculation of a posteriori probabilities. We show that the *Deletion Principle* and the *Conditional Ratio Shortcut* can lead to wrong results, because they are based on the assumption that the probabilities of the remaining feasible hypotheses of cards retain their relative ratio after the evidence has become available. Finally we present an amended deletion concept that applies also in cases where the probabilities of the remaining hypotheses do not retain their a priori ratio.

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2 The *Vacant Places Method* for calculating a priori card distributions

Classic probability theory provides a formula for the probability that n cards are split in k cards and $n-k$ cards between opponents West and East. In particular the probability that West has k cards and East has $n-k$ cards out of a set of n cards in a suit is

$$P(k, n-k) \ = \ \frac{\binom{n}{k}\binom{26-n}{13-k}}{\binom{26}{13}} \tag{1}$$

The method of vacant places [Vp] gives an alternative way for calculating these probabilities. Usually the reasoning starts with considering the ways in which two cards can be split between the two opponents. One card is dealt arbitrarily to one of the opponents, say West. Then for the second card there are 12 vacant places left with West and 13 vacant places left with East. Consequently (given that West has that first card) the probability for a $1-1$ split from this deal is $\frac{13}{25} = .52$ and the probability for a $2-0$ split from this deal is $\frac{12}{25} = .48$. Since West is dealt that first card with probability .5, the conditional probabilities are .26 and .24 respectively. From a symmetry argument then follows $P(2,0) = P(0,2) = .24$ and $P(1,1) = .52$ in agreement with classic probability theory.

To calculate the probability of e.g. a $1-2$ split a weighted average is taken of the probabilities of the $1-1$ split and the $0-2$ split, the weights coming from applying the Vacant Places Method. Thus

$$P(1,2) = \frac{12}{24}P(1,1) + \frac{13}{24}P(0,2) = .39$$

again in agreement with classic probability theory. By using an inductive reasoning, we will now formally prove that the method of vacant places for calculating these a priori probabilities of a suit split is in agreement with classic probability theory.

Lemma 1 *Let $P(k, n-k)$ be the probability that n cards are split in k cards with West and $n-k$ cards with East according to classic probability theory as given in equation 1.*
Let $Q(k, n-k)$ be the probability that n cards are split in k cards with West and $n-k$ cards with East according to the inductive application of the method of vacant places. Then for all $n \geq 0$ and $0 \leq k \leq n$ $\;P(k, n-k) = Q(k, n-k)$

Proof It is easily verified that $P(0,0) = Q(0,0) = 1$. Suppose that for fixed n and $0 \leq k \leq n$ $P(k, n-k) = Q(k, n-k)$. The method of vacant places calculates $Q(k, n+1-k)$ according to

$$Q(k, n+1-k) = \frac{13-(k-1)}{26-n}Q(k-1, n-k+1) + \frac{13-(n-k)}{26-n}Q(k, n-k)$$

and in case of $k = 0$ (in a similar way for $k = n+1$)

$$Q(n+1, 0) = \frac{13-n}{26-n}Q(n, 0)$$

Applying this expression for $Q(k, n+1-k)$ and using $P(k, n-k) = Q(k, n-k)$ we find

$$Q(k, n+1-k) \;=$$

$$\frac{13-(k-1)}{26-n} \frac{n!\,(26-n)!}{(k-1)!\,(n-k+1)!\,(13-k+1)!\,(13-n+k-1)!\,\binom{26}{13}} +$$

$$\frac{13-(n-k)}{26-n} \frac{n!\,(26-n)!}{k!\,(n-k)!\,(13-k)!\,(13-n+k)!\,\binom{26}{13}} =$$

$$n!\left(\frac{k}{k!(n-k+1)!} + \frac{1}{k!(n-k)!}\right)\frac{(26-n-1)!}{(13-k)!(13-n+k-1)!}\frac{1}{\binom{26}{13}} =$$

$$\frac{\binom{n+1}{k}\binom{26-(n+1)}{13-k}}{\binom{26}{13}} = P(k, n+1-k).$$

which proves this lemma. □

We conclude that although the *Vacant Places Method* uses another reasoning compared with classic probability theory, the results for the a priori distributions of cards in a suit are the same.

3 The *Vacant Places Method* for calculating a posteriori card distributions

The *Vacant Places Method* is sometimes advocated for calculating the suit split probabilities after certain evidence has become available.

Rubens [Ru] p99 calls it *The Places-Open shortcut ... If West has **a** places in his hand that could be filled with unknown cards to East's **b** such places, the ratio of West's chance of holding a particular unlocated card to East's chance is* $\frac{a}{b}$ ... and although this definition poses no restriction on the use, he proceeds with examples of a priori deals. The evidence can be a result of the bidding, the lead or the way opponents follow suit during the play. Application of the *Vacant Places Method* is not straight forward when calculating a posteriori probabilities i.e. one cannot simply compare the vacant places at the critical moment, which becomes immediately clear in the following examples.

- Suppose that declarer in a NT contract has found out that West has started with 5 cards in a color and East had originally 3 cards in that color. If there is a dual option for finessing a Q in another color, should this evidence then indicate that finessing over East is the best line of play? In spite of Rubens' definition, the answer is no! Most players lead their longest suit so the choice of the lead color is not random, while the vacant places method is based on that very assumption. This is what Philip Martin [Ma] calls the Monty Hall trap as demonstrated in the three doors problem.

- When four cards $Qxxx$ are missing, playing for the drop is the best line of play. This could (wrongly) be understood by applying the *Vacant Places Method* in the second round at the moment when West has shown two small cards and East has shown one small card. Then East has one more vacant place, so playing for the drop is better than finessing. Now take the case where five cards $Qxxxx$ are missing. Analog application of the *Vacant Places Method* leads to the same (wrong) conclusion that playing for the drop is better than finessing, while from the a priori distribution it is known that finessing is better.

There are more complex problematic positions for the use of this simplification. Kelsey [Ke] p79 considers the position

♠AJx

□

♠KTx

where there is a two-way finessing position. It is expert play to delay the finesse position until some count of the defender's hand is available. Kelsey states ... *Suppose you eventually discover that West must have started with five ♠ and East with two ♠* ... than it is wise to finesse through West after playing the K in the first round. Kelsey proceeds with *... Nor does it make any difference if West has discarded a couple of ♠♠ on other suits. It is the number of cards dealt to each defender that matters, not the number each holds at the critical moment ...* This is the same argument as used by Reese [Re] p40. [1]

This example indicates that you cannot simply use the *Vacant Places Method* at the critical moment, which is true. However Kelsey's statement that all that matters is the number of cards dealt is imprecise. During bidding or play there must have shown up evidence for the conclusion that W has started with 5 ♠♠. This evidence influences the a posteriori probability of all remaining hypotheses of the initial card distribution. So not the sole number of cards that have been dealt to W counts. This should be the a posteriori (based on the evidence) probability for West having been dealt with that number of cards.

The importance of knowing the complete distribution of a suit is expressed by Kelsey as a rule *When the distribution of one suit (or more) is completely known, the probability that an opponent holds a particular card in any other suit is proportional to the number of vacant places remaining in his hand.*

Although it is not explicitly said by Kelsey, this rule suggests that in absence of complete knowledge no inference about the distribution in another suit can be made. Suppose there is evidence that West has 4 or 5 cards in a suit and East has 2 or 1 card in that suit. Based on each assumption the *Vacant Places Method* could be applied. If in both cases the same inference on the place of a critical card follows (which is in general true in the absence of any other evidence), then this inference is valid although the distribution of the first suit is not completely known.

The rule also suggests that the knowledge must refer to the number of cards in a suit. Suppose West has lead a K in another suit and the declarer must again decide on the finesse in the AJx

[1] When explaining the so-called coin test at the end of his chapter on Restricted Choice, Reese writes
Many readers will find some of the conclusions in this chapter hard to believe. Believing that the odds change with every card played, they will see no advantage in going back, as it were, to study the a priori expectations. To dispel that illusion it may help to make a simple experiment in a medium other than cards.
Suppose that there are five coins, four heads and one tail. They are divided into two piles, three on the left and two on the right. now you would say that it was $3:2$ against the tail being included in the smaller pile. Now take two coins away from the larger pile, with the proviso that neither of them being the tail. (That is what happens in bridge, where the discarding is selective and a player who has the critical honour does not play it wantonly.) At this stage there is only one coin on the left and, as before, two on the right. It remains $3:2$ against the tail being on the right.
This experiment cannot be questioned, however it is not always a valid representation of the carding in bridge. The evidence in Reese's experiment, two heads are taken from the left pile has the same probability under both hypotheses i.e. he tail being on the left or on the right. This is not necessarily so in bridge, as is explained in this article.

against KTx position for the ♠. Under the reasonable assumption that West holds the Q when he leads the K of a suit, the odds are for a finesse through East for the ♠Q.

In both cases all that matters is the a posteriori probability for the placing of that card. Let $\{H_i\}$ be a set of alternative hypotheses with a priori odds $P(H_i)$ and let A be some evidence. The a posteriori odds $P(H_i/A)$ for these hypotheses can be calculated with Bayes' rule

$$P(H_i/A) = \frac{P(A/H_i)}{\sum_j [P(A/H_j)P(H_j)]} P(H_i) \tag{2}$$

We conclude that the *Vacant Places Method* should in general not be applied for the calculation of a posteriori distributions. Other simplifications have been suggested for that purpose. These simplifications delete the hypotheses which are ruled out by the evidence and they assume that the ratio of the odds for the hypotheses that are not ruled out by the evidence, remain the same.

4 The structure of simple deletion rules

The *Deletion Principle* is suggested by Kelsey [Ke] when calculating a posteriori probabilities. Kelsey states his *Deletion Principle* as *When the opponents follow to the play of a suit with insignificant cards, the impossible distributions are deleted and the probabilities of the remainder retain their relative magnitude.* The *Conditional Ratio Shortcut* of Rubens [Ru] is advocated for the same situation. It is stated as *When one or more splits of a suit are eliminated, the chances of those remaining retain the same ratios.* The two definitions are not identical. Kelsey requires that only insignificant cards are being played by the opponents, while Rubens does not. What significant cards are is defined by Kelsey ... *significant in the sense that a defender would never play it, unless forced to do so...* [2] Let us use classic probability theory to see what actually happens with a posteriori probabilities when some splits are eliminated.

Lemma 2 *Let $\{H_i\}$ be a set of alternative hypotheses for the distribution of outstanding cards in a suit with a priori probabilities $P(H_i)$. Let E be evidence with rules out some of these hypotheses. The ratio of the a posteriori probabilities for the remaining hypotheses remains the same if and only if $P(E/H_m) = P(E/H_n)$ for all remaining hypotheses H_m and H_n.*

Proof According to Bayes' rule we have for all a posteriori probabilities

$$P(H_i/E) = \frac{P(E/H_i)}{P(E)} P(H_i) = \frac{P(E/H_i)}{\sum_j [P(E/H_j)P(H_j)]} P(H_i). \tag{3}$$

For a hypothesis that is ruled out, we have $P(E/H_i) = 0$ and consequently $P(H_i/E) = 0$. For the ratio of two arbitrary remaining hypotheses we find

$$\frac{P(H_m/E)}{P(H_n/E)} = \frac{\frac{P(E/H_m)}{P(E)} P(H_m)}{\frac{P(E/H_n)}{P(E)} P(H_n)} = \frac{P(E/H_m)P(H_m)}{P(E/H_n)P(H_n)}, \tag{4}$$

from which follows that a sufficient and necessary condition for retaining the ratio between hypotheses is that $P(E/H_m) = P(E/H_n)$ which proves this lemma. □

[2] This definition raises some questions. If a defender holds just one small card, he is forced to play it. If a defender holds a doubleton JQ he is not forced to play a particular card. A defender may false card, which in essence is the play of a significant card when not forced to do so. Clearly when EW are holding $234JQ$, the 234 are insignificant cards and the JQ are significant cards.

5 Examples of the *Deletion Principle* and the *Conditional Ratio Shortcut*

When these simplification concepts are applied in a particular example, the necessary and sufficient condition of equal evidence probability is not always fulfilled. If this condition holds then the application is justified and the results are correct. In other cases the results are in general incorrect.

Consider as a first example

$\spadesuit KTxx$

$\square$

$\spadesuit AJxxx$

with $Q432$ missing in that suit. Without any further information the a priori odds that the Q is in either opponents' hand are equal. The opponents play the 2 and the 4 to the play of the A by declarer. West plays the 3 to a small card of declarer to the K. It is common knowledge that in this case playing for the drop is superior. How does this follow from the Deletion Principle? The actual play rules out all but two hypotheses about the distributions of the four cards $Q23-4$ with a priori probability .062 and $23-Q4$ with a priori probability .068. Under both hypotheses the chance of the actual play is the same, namely 1. Then the Deletion Principle states that the a posteriori probabilities have the same ratio and become .46 and .54. Playing for the drop is better than finessing. Note that playing insignificant cards here results in equal chance for the evidence under all remaining hypotheses.

Consider as a second example

$\spadesuit Kx$

$\square$

$\spadesuit AJxxx$

as discussed by Rubens [Ru] p108. In the first round to the K both defenders play a small (insignificant) card. In the second round the J is covered by West's Q. What is the probability that the remaining small cards are distributed $1-1$?

Let us assume that $2345Q$ is with EW, which means that EW can only play insignificant cards next to the Q. (We leave the situation where the T is with EW undiscussed.) It now comes down to the assumption how and when defenders play the Q. Under the reasonable assumption that the declarer needs more than 3 tricks in the suit, West can afford to hold up the Q and explert play is to do so once and a while. Let p_i be defined as $p_i = P(E/H_i)$ which is the probability that the Q is played by W in the second round. The a priori and a posteriori probabilities of the $2-4$ and $3-3$ hypotheses are given below

Hyp	Split	P a priori	Pr(E/hyp)	P a posteriori
H_2	$Vxxx-xx$	.162	p_2	$\frac{.162p_2}{.162p_2+.259}$
H_3	$xxxx-Vx$	.081	0	0
H_4	$Vxx-xxx$	.178	1	$\frac{.178}{.162p_2+.259}$
H_5	$xxx-Vxx$	.178	0	0
H_6	$Vx-xxxx$	.081	1	$\frac{.081}{.162p_2+.259}$
H_7	$xx-Vxxx$	.162	0	0

Without an explicit assumption on the defense strategy, all that can be said about the a posteriori odds of a $3-3$ distribution is that the probability is in the range $.42-.69$. Only if $p_2 = 1$ (always covering the J), the ratio of probabilities for the hypotheses H_2 and H_4 is not changed by the evidence and applying the conditional ratio shortcut is justified. A serious flaw in the definitions of the *Deletion Principle* and the *Conditional-Ratio Shortcut* is the absence of an assumption on defenders strategy. The implicit reasoning that the defense strategy is unimportant as long as only insignificant cards are played is not always sufficient. This is illustrated by the following examples.

Kelsey's [Ke] p68. applies the *Deletion Principle* when discussing the following examples.

$AKQxx$

□

xx

When the opponents follow suit with insignificant cards, then according to Kelsey the a posteriori probabilities of the suit being divided $3-3$ and $2-4/4-2$ become .423 and .577, which is the same ratio as for the a priori probabilities .355 and .484. Kelsey concludes ... *this is a significant increase, but it still can't compete with the fifty-fifty chance of a finesse* ... in another suit. Kelsey's reasoning is correct because in his example opponents have $6789TJ$. Now all cards are the same and EW can play randomly.

In general EW do not know which cards they together hold. They see the table with $AKQ53$ and must make a judgement what cards are significant. We now assume that the T and J are seen as significant cards by EW in the sense that defenders will not show such a card unless forced to do so. Let us calculate what the odds are from classic probability theory. Both cards are denoted as h for honor in the table below. Further we define the event E as both opponents follow with insignificant cards as is done by Kelsey. insignificant cards are denoted as x. We only distinguish between all relevant hypotheses.

Hyp	Split	P a priori	Pr(E/hyp)	P a posteriori
H_2	$hhxx-xx$	.096	1	.235
H_3	$hxxx-hx$	.128	0	0
H_4	$xxxx-hh$	.017	0	0
H_5	$hxx-hxx$	.216	1	.53
H_6	$hhx-xxx$	.070	0	0
H_7	$xxx-hhx$	.070	0	0
H_8	$hh-xxxx$	.017	0	0
H_9	$hx-hxxx$	.128	0	0
H_{10}	$xx-hhxx$	.096	1	.235

We see that, due to our alternative definition of significant cards, hypotheses H_6 and H_7 are also ruled out by the actual play. The only remaining hypotheses are H_2, H_5 and H_{10}. As a consequence the application of the *Deletion Principle* leads to a posteriori probabilities of .235, .53 and .235 respectively. Now playing for the drop is better than a fifty-fifty chance in another suit.

The assumption that $P(E/H_i)$ is the same (namely 1) for all remaining hypotheses can be questioned. Experts in East holding $JTxx$ could drop the J or T in the second round. This not only points to a weakness in the definition of a significant card (such a card can be false carded on purpose), it also shows that $P(E/H_i)$ can have different values for the remaining hypotheses. Assume for the sake of argument that an expert East plays x in the first round and at random in the second round. Hence in the example above $P(E/H_{10}) = .333$ which changes the ratio of the probabilities from a priori to a posteriori. In such cases application of the *Deletion Principle* is unjustified.

Kelsey discusses a second example

$AKQTx$

□

xx

Again, when both opponents follow suit with insignificant cards, then the a posteriori probabilities of the suit being divided $3-3$ and $2-4/4-2$ change in an different way. The reason is that now only the J is a significant card. All $3-3$ distributions remain feasible and the actual play has the same probability under all hypotheses. Hence the application of the *Deletion Principle* is justified and the results are in agreement with classic probability theory. All $3-3$ distributions remain feasible and the actual play has the same probability under all hypotheses. Hence the application of the *Deletion Principle* is justified. The a posteriori probabilities the suit being divided $3-3$ and $2-4/4-2$ now become .524 and .476. This is in agreement with classic probability theory, as can be seen in the table below.

Hyp	Split	P a priori	Pr(E/hyp)	P a posteriori
H_2	$Jxxx-xx$	.161	1	.235
H_3	$xxxx-Jx$	.081	0	0
H_4	$3-3$	.355	1	.53
H_5	$Jx-xxx$	.081	0	0
H_6	$xx-Jxxx$	.161	1	.235

In conclusion we can say that application of the *Deletion Principle* and the *Conditional-Ratio Shortcut* is sometimes incorrect. As they are formulated they are incorrect. If and only if $P(E/H_m) = P(E/H_n)$ for all the remaining hypotheses, both deletion rules lead to the right results.

6 An alternative Deletion Principle

It is natural to ask for a Deletion Principle that takes into account the drawbacks discussed in this article. The most simple answer is that such a principle is in fact Bayes' rule. However knowing that many hypotheses will be ruled out by the evidence of card playing by opponents, there is room for a RDP (acronym for Right Deletion Princple) . This reads

When the opponents follow to the play of a suit, the impossible distributions are deleted and the probabilities of the remaining distributions must be multiplied with the probability of the actual play when that distribution applies. Then and only then the a posteriori probabilities of the remaining distributions are calculated correctly. The a posteriori ratio of two remaining hypotheses may differ from the a priori ratio.

Let us consider one more example

$AKQ9x$

□

xx

Again, when both opponents follow suit with insignificant cards, then the a posteriori probabilities of the suit being divided $3-3$ and $2-4/4-2$ change in an different way. The reason is that now the J and the T are significant cards. All $3-3$ distributions remain feasible however the actual play has different probability under all hypotheses. The reason is false carding by an expert East when he holds $JTxx$. Hence the application of the *Deletion Principle* is not justified, as can be seen in the table below

Hyp	Split	P a priori	Pr(E/hyp)	P a posteriori
H_2	$xx - JTxx$	.097	p_2	$\frac{.097p_2}{.097p_2+.311}$
H_3	$Tx - Jxxx$	.065	0	0
H_4	$Jx - Txxx$	.065	0	0
H_5	$JT - xxxx$	.016	0	0
H_6	$xxx - JTx$	.071	0	0
H_7	$Txx - Jxx$	.107	1	$\frac{.107}{.097p_2+.311}$
H_8	$Jxx - Txx$	.107	1	$\frac{.107}{.097p_2+.311}$
H_9	$JTx - xxx$	.071	0	0
H_{10}	$xxxx - JT$	.016	0	0
H_{11}	$Txxx - Jx$	.065	0	0
H_{12}	$Jxxx - Tx$	.065	0	0
H_{13}	$Jtxx - xx$	.097	1	$\frac{.097}{.097p_2+.311}$

By making an additional assumption on the style of false carding under H_2, e.g. $p_2 = .5$, one finds the correct a posteriori probabilities. This is taken into account by the *Right Deletion Principle*. Only if we assume that East never false cards under H_2, then the a posteriori probabilities retain their relative magnitude.

7 Conclusions

- Classic probability theory produces the correct probabilities for a priori card distributions in bridge. The *Vacant Places Method* is an alternative (simpler) way to calculate these.
- Bayes' rule produces the correct probabilities for a posteriori card distributions in bridge. The *Deletion Principle* and the *Conditional Ratio Shortcut* produce the same results if and only if the evidence has the same probability under all remaining hypotheses.
- The condition that the defense plays only insignificant cards is insufficient to guarantee this requirement of equal evidence probability.
- It is possible to define a *Right Deletion Principle* that produces the correct a posteriori probabilities. In essence this RDP is a fracture simplification of Bayes' rule.

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